Infinite Sums and Reaching Infinity

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IB DP Mathematics SL Internal Assessment

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Research Question:

Does the result of the infinite sum 1+2+3+4+5+... equal $\frac{-1}{12}$ –as is claimed in string theory–or is it completely incorrect?

Personal Engagement:

I have been interested in number theory and string theory ever since I listened to a speaker in the Distinguished Speaker Series at Texas A&M Corpus Christi. David Gross and Adam Reiss both spoke of their involvement in string theory and how they apply it to the world around us on a molecular scale. Not only for their work in space exploration, but also for the research on matter and antimatter. It was really fascinating and upon further personal research I came upon a video which described an infinite series considered true in string theory. However, the video did not go as in-depth as would have been preferred. When searching for another video that could potentially explain the process with more detail, I stumbled upon a different video claiming that anything showed in the original video was in fact it was completely wrong. This piqued my interest as I couldn't imagine something being the basis of string theory to be incorrect. Then I once again encountered the solution to this sum while doing research for my Extended Essay because it was about Indian mathematicians, and so I read about the kind of work that Srinivasa Ramanujan was involved in. It revolved heavily around number theory and even made an equation for infinite series called the Ramanujan Summation. From this I also saw the calculations he made during this research in order to prove many infinite series, two of which are going to be presented in this paper. And so from all of this knowledge on the subject, I wanted to see just how incredible these infinite series are and how these brilliant mathematicians went about solving them.

Introduction:

The proof of this infinite series is actually so important as a basis for understanding string theory that they put it in the beginning of books to learn string theory.[1] However the validity of this series has been placed under some scrutiny by other mathematicians and this is completely viable because the whole thing is after all, just a "theory." This piqued my interest because string

[1] N. (2014, January 09). ASTOUNDING: 1 2 3 4 5 ... = -1/12. Retrieved March 26, 2018, from https://www.youtube.com/watch?v=w-I6XTVZXww

theorists and theoretical physicists working on string theory are considered to be some of the most brilliant minds, and for them all to believe that the sum of 1+2+3+4+5+... does in fact equal -1/12 means there must be a logical/mathematical explanation for it. Through this internal assessment I will be looking into the methods used to prove this answer, as well as the logic and thought process used to disprove it. This will give me a better understanding of what infinite series consist of and what sort of mathematical methods are utilized in proving these sums. I will then analyze the strengths and weaknesses behind each argument made and then pick a stance on whether or not that infinite sum has a valid answer. This will also show me that even in mathematics there are many things up to discussion and can be proven true or false based on the kind of process involved and what the answer is claimed to represent. In this case, the answer is supposed to show a solution to the summation of all real numbers.

Infinite Sum:

$1 + 2 + 3 + 4 + 5 + \ldots = \frac{-1}{12}$

Proof that 1+2+3+4+5+... does equal -1/12

Before finding the sum of the final infinite series, there needs to be a consideration of two other important infinite sums.

The first one (S_1): 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + ...

The second one (S_2): 1 - 2 + 3 - 4 + 5 - 6 + ...

In order to see how these two can be used to solve for the original infinite sum, each one must be evaluated individually to yield a result that can then be used to derive the last infinite sum.

First Infinite Sum

There are two way that the terms in the first sum can be grouped, each with a different solution:

(1-1) + (1-1) + (1-1) + (1-1) + … = 0

1 + (-1+1) + (-1+1) + (-1+1) + … = 1

However the fact that there are two distinct solutions doesn't help to find the one answer that satisfies the infinite solution. In fact, if this is the case then it can be classified as a "divergent series." This means that the values don't approach a certain value and either grow further apart or just never meet.

If this infinite sum were to be given the general label of S_1 then we can try a different method of finding the solution, let's see what happens when we just subtract S_1 from one on both sides:[2]

$1 - S_1 = 1 - (1+1-1+1-1+...)$

$= 1-1+1-1+1-1+...$

$= S_1$

$1 - S_1 = S_1$

$1 = 2S_1$

$½ = S_1$

The significance of this particular solution to the infinite sum is that it lies in between the two other solutions and so in most cases ½ is considered to be the real solution as it is basically the average of the other two answers. It is important to note that the other two solutions are not to be disregarded. They are just not used to express an answer to this infinite sum.

There are other ways to prove this infinite series, some using a different mathematical approach than the others but all ending up with the answer of ½.

$S_1 = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

= 1 0 1 0 1 0 ← Partial sums

= 1 ½ ⅔ ½ ⅗ ½ ← Average of partial sums

[2] N. (2013, June 25). One minus one plus one minus one - Numberphile. Retrieved March 26, 2018, from https://www.youtube.com/watch?annotation_id=annotation_2443680779&feature=iv&src_vid=w-I6XTVZXww&v=PCu_BNNI5x4

The partial sums are the values found when the infinite series is stopped at that point. So in the first spot the sum is 1, but in the second spot the value is 1-1 so the partial sum is 0. And it just continues as such. From the calculation above, it can be seen that every other term equals ½ and all the other terms are converging on ½ so even though the original series can be defined as "divergent" because the values do not approach one number, this process reveals a new way to perceive it as "convergent." *Figure 1-1* on the next page provides a visual representation for this relationship in the numbers. There is a special term for this sort of series and that is known as a "Cesàro Convergent."[3] This is opposed to being standard convergent in which the series itself converges on a single value. A series is Casàro Convergent when the sum itself does not converge on a value, but the "arithmetic mean" of those values converges on a value.[3] It was given this name because of the mathematician that came up with it, Ernesto Cesàro.

Sample Calculation for Arithmetic Mean of Partial Sums:

Numbers: 1 0 1 0 1 0

Averages: 1 ½ ⅔ ½ ⅗ ½

Primarily what is being done is that all the terms are being added until the n^{th} term and then divided by n number of terms to get the average of the partial sums.

$$\frac{1}{1} = 1$$

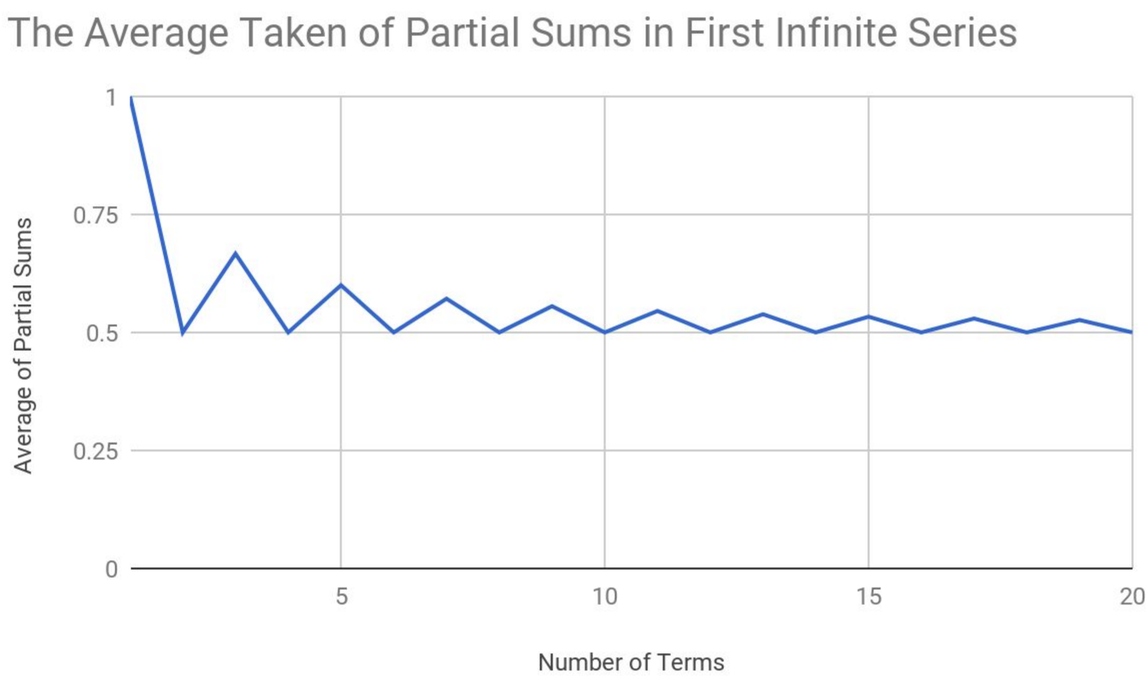
$$\frac{1+0}{2} = \frac{1}{2}$$

$$\frac{1+0+1}{3} = \frac{2}{3}$$

$$\frac{1+0+1+0}{4} = \frac{2}{4} = \frac{1}{2}$$

[3] M. (2016, April 22). Ramanujan: Making sense of 1 2 3 ... = -1/12 and Co. Retrieved March 26, 2018, from https://www.youtube.com/watch?v=jcKRGpMiVTw

Figure 1-1: Partial Sum Averages for First Infinite Series



In *Figure 1-1* it can be seen that the arithmetic mean of the values do appear to converge. Eventually they are flattening out to reach 0.5 or ½. This graph is actually an excellent portrayal of the Cesàro Convergent. The path of the graph can be compared to that of a ball after it it dropped, where it keep bouncing up, but each time lower than the last, until it stop bouncing and stays at the height of the ground, the same thing happens in *Figure 1-1* with each subsequent average taken.

Second Infinite Series

The answer for this infinite sum actually relies on going back to the solution for the previous sum and substituting it into this infinite sum.[1] This sum will be labelled as S_2:

$$2S_2 = S_2 + S_2$$
$$= 1 - 2 + 3 - 4 + 5 - 6 + \ldots$$
$$\underline{+ \quad 1 - 2 + 3 - 4 + 5 - \ldots}$$
$$2S_2 = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

The series 1 - 1 + 1 - 1 + … was labelled as "S_1" and so it can be placed there instead

$2S_2 = S_1$

The solution to S_1 was ½ and therefore,

$2S_2 = ½$

$S_2 = ¼$

And so with this method of finding the solution to the infinite sum, it can be seen that with the use of S_1, the value for S_2 can be found using a logical equation. The second time the equation is written down in the addition problem it is just shifted one space to the right which is right in this infinite sum because it is the equivalent of adding a "+0" which has no value and so does not affect the result of the infinite sum.[4]

Even with this infinite sum, the "Cesàro Convergent" can be applied in order to reach the same solution of ¼.

$S_2 = 1 - 2 + 3 - 4 + 5 - 6 + \dots$

= 1 -1 2 -2 3 -3 ← Partial sums

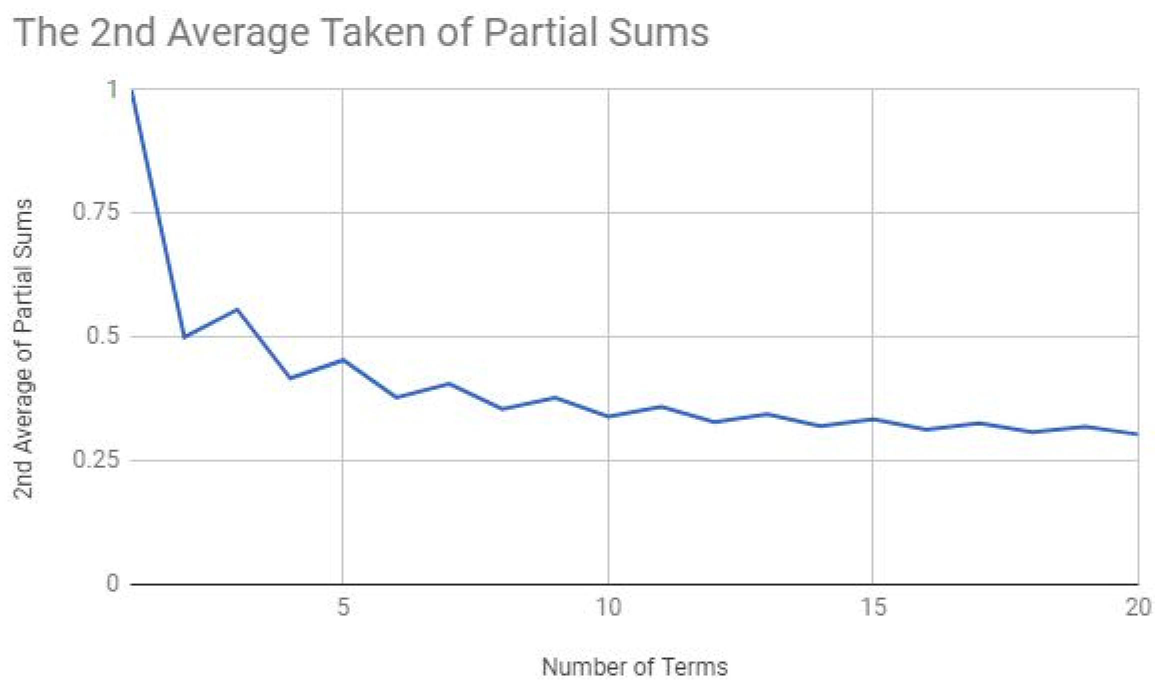
= 1 0 ⅔ 0 ⅗ 0 ← 1st Average of partial sums

= 1 ½ 5⁄9 5⁄12 34⁄75 17⁄45 ← 2nd Average of partial sums

In this case the first set of averages did not yield any sort of convergence, however according to the method employed by mathematicians with the Cesàro Convergent, it is probable to take as many averages as necessary to find a convergence. The second average yielded a group of values which could be seen as getting closer and closer to ¼ and that meant the limit was approaching that value. This relationship is not that obvious to see and so *Figure 1-2* on the next page helps to visualize how the values slowly seem to approach "0.25" which is ¼. That satisfies the criteria for it to be considered a Cesàro Convergent.

[4] M. (2018, January 13). Numberphile v. Math: The truth about 1 2 3 ...=-1/12. Retrieved March 26, 2018, from https://www.youtube.com/watch?v=YuIIjLr6vUA

Figure 1-2: Second Partial Sum Averages for the Second Infinite Series



As previously explained, the Cesàro Convergent allows for multiple arithmetic means to be taken in order to see if there is any point at which the series will become convergent. In the case of this series, it only took two averages to see that the values actually became convergent. This means that the rules normally applied to convergent series, can also be used in order to solve this series as will be shown below. This serves to further prove why the method employed to solve this series is correct, and has mathematical reasoning behind it.

Final Infinite Series

The simplest way−one of many ways−to solve this infinite series is as follows:

If the solution to 1+2+3+4+5+... is seen as some constant "S" then it is okay to write it as

$S = 1 + 2 + 3 + 4 + \dots$

Subtracting four times the infinite series "S" will yield (assuming this series is also convergent means that the shift occurring in the second line is also allowed because it follows the rules of convergent summations)[4]:

$$\begin{aligned} S &= 1 + 2 + 3 + 4 + \dots \\ -(4S &= \quad\ 4 + \quad\ 8 + \dots) \\ \hline -3S &= 1 - 2 + 3 - 4 + \dots \end{aligned}$$

This infinite series on the right is familiar, and can be identified as the series labelled "S_2"

$-3S = S_2$

Since the solution of S_2 was found to be ¼ that can be substituted in the formula to solve for "S"

$-3S = ¼$

$S = (¼) / (-3) = (¼) * (-⅓)$

$S = \frac{-1}{12}$

Logic Behind Proving the Infinite Sum as False

There is not much to the reason behind this claim being false. To put it out there simply, it is not possible. There is no mathematical way−at least one that makes sense−that allows for the sum of all positive real numbers to every yield a negative answer, let alone $\frac{-1}{12}$. The reason mathematicians have been receiving this as their answer, is because they are considering this function to be something it is not. None of the aforementioned series are convergent−meaning they all eventually reach one value−but are divergent.[4] This means that all the techniques used in the proofs to show the solutions for these series cannot be applied because those rules are for convergent series. To add onto this, just because the arithmetic mean of the values in the sum are convergent, does not automatically translate to the series beinvergent and so the same rules and processes cannot be employed.[5] Therefore, despite everything that has been said about different assumptions being made and rules being followed, the answer of $\frac{-1}{12}$ is incorrect. In a more common logical approach, even a layman could see that 1 + 2 + 3 + 4 + 5 + … actually approaches infinity (∞) and thus cannot have a numerical value assigned to the solution.

Conclusion

I have to say that although one answer makes the most logical sense in the realm of math and seems the normal way to approach it, there is nothing normal about string theory. String theory is all about what is not normal and what we humans are not used to interacting with. And in order to quantify the things that happen at that level without a complete knowledge, assumptions have to be made and that is exactly what these mathematicians did. They made knowledgeable assumptions and based all their calculations on that. So to judge their answer with standard math conventions would be unreasonable and obviously lead to the conclusion that the answers are wrong. But I believe that as long as the method of reaching the answer are consistent, both solutions can be considered correct. For someone taking a math test in their Calculus class after learning about infinite series, they should most definitely put +∞ as the answer to 1+2+3+4+5+... However to a string theorist trying to wrap their head around the way things work on the molecular level, they will be using the $\frac{-1}{12}$ quite frequently. It all depends on the perspective. Of course there are many implications to this investigation and its scope reaches far beyond just this simple infinite series. The applications reach into string theory and how things work when at such small scales.

As with every investigation, there are always weaknesses that have to be identified as the source of some limitations in the explanation and exploration of a topic or concept. In this case, the limitation would be the resources available to the student to find information. There were videos that explained the notes that Ramanujan took in order to explain these mathematical phenomena, but there were not primary sources that actually showed them. Of course there was plenty of information in these videos, but a book with the said infinite sums being explained

[5] Peny, K. (2015, October 26). Response to Numberphile's ASTOUNDING 1 2 3 4 ... = minus 1/12 (sum of natural numbers to infinity). Retrieved March 26, 2018, from https://www.youtube.com/watch?v=BpfY8m2VLtc

more in depth would have been invaluable. Unfortunately, books like those can go well over $100 and that makes them too expensive to afford. These videos had to be taken as an acceptable substitution. Despite this obvious limitation to the overall investigation of these infinite sums, however, the resources that were available had a lot of information present that was valuable to the overall understanding of concepts. There was plenty there that helped both to prove, and to disprove the sides of the argument being discussed and the perspectives helped to create amore complete image of what exactly was going on and what to make of it in the end of the research.

In addition to what was discussed in this Internal Assessment, there are many ways this could be expanded to include other areas of mathematics and physics. The importance of infinite series in string theory is quite significant. It is used in the calculations of many values and the only way to find out would be to look further into string theory. It has been said that the value for 1+2+3+4+5+... comes from the Riemann Zeta Function and that this function has monumental importance in string theory.[6] Therefore, the best way to expand upon the knowledge gained through this assessment would be to investigate the use of this series in the realm of string theory and how exactly it has impacted those ideas and concepts.

Summary

This investigation has revealed a lot about the nature of these functions and how many differing views can exist even within the field of mathematics. In the end it could be seen that there were many ways to solve this infinite series, and also many ways to prove that there is no singular answer to it. Either way, it seems that there are assumptions being made and rules being followed in order to reach both conclusions and both can be seen as possible and plausible given the circumstances under which they were found. Overall, there is no absolute correct answer, it depends on the terms of the question and situation under which it is being solved to find what answer is being looked for. The solution to 1+2+3+4+5… could actually be $\frac{-1}{12}$.

[6] Rosario, T. Links between string theory and the Riemann's zeta function. Retrieved March 26, 2018, from http://empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/nardelli2010a.pdf

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