Identifying Fibonacci Numbers and Sequences Through Bach's Composition

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Preface:

Music has been around me my whole life, and mathematics was always involved whether it is counting or analyzing the rhythm. It creates harmony, and there are foundations that creates it. It is about rhythm and melody, and the changing of notes in relation to time. Arithmetic and geometric patterns can be examined in a typical composition of music. In this paper, utilizing mathematical correlations, analysis are drawn. Through examination of the selected pieces, such as *Fugue of Art* and *Well Tempered Clavier* No.1, which serve to dictate the form of pieces in a subtle manner. Furthermore, Bach creates a rhytimetic climax on certain measures corresponding to the Golden Ratio. It is also present and is through the composer's choice of notes. Utilizing these mathematical connections, not only it establishes a deeper correlation between music, but also provides a beauty of Fibonacci sequence. Furthermore, this indicates that Math is not only about numbers, but also everywhere one sees.

Leonardo Fibonacci, a mathematician from Pisa, he developed a mathematical theory which establishes an infinite series of integers. His sequence begins with the numbers followed by 1. Each successive term is constructed by adding the previous terms. A Fibonacci sequence represents a neurological sequence that reflects the geometric harmony found in nature. From sunflower to a lotus, the possibilities are everywhere. Employed in mathematics as far as back in the first century, it epitomizes the golden ratio, a balance of proportions found throughout not only with music, but also with architecture and art as well. A sequence of Fibonacci ratios is the series of numbers produced when each Fibonacci is divided by the number that precedes it.

The golden ratio is a concept that is also found in music. It is often used to generate rhythmic change or to develop a melody line, and is found in the musical timing of compositions. As a musician, I always wanted to understand the mathematical aspects of musical notes, which is found in all of the instruments. In this paper, I will be investigating the musical notes, the intervals through the ratios of Fibonacci sequence through the snippets or excerpts from Bach’s compositions. Furthermore, by discovering, I will demonstrate how the golden ratio affects his musical composition.

Analysis:

Connections between the past and the present, between math and music, and other subjects, is possible to get discovered in an unlikely manner. Who would think that a link could be drawn from an Baroque era musician, by means of arithmetic and a ratio?

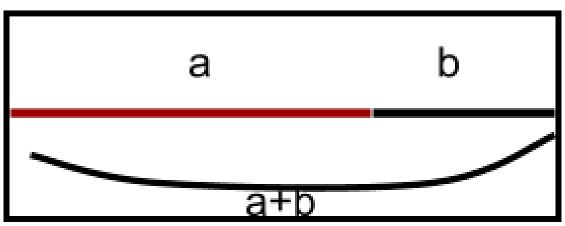
Before answering that question, the definition and significance of both the golden ratio and Fibonacci numbers must be made.

Also known as the golden section or golden mean, is first defined through the following equation:

$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

Observing the formula above, it is stated as “A plus B is to A as A is to B”. This indicates that the total length of *a* plus *b* is to the “longer segment (A) as A is to the shorter segment (B)”.

For visualization:



The Fibonacci sequence, a structure that shows $x_{n+2} = x_{n+1} + x_n$ (n=1), allows to generalize the analysis of Bach’s compositions.

If we let $x_1 = a$, and $x_2 = b$, an illustration is depicted as:

a, b, a + b, a + 2b, 2a +3b, 3a + 5b, … (sequence 1)

It starts with a =1 and b = 1. The first 16 terms are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987…

Because of that, it is linked to the irrational number, called the golden ratio:

$$\Phi = \frac{1+\sqrt{5}}{2} \cong 1.618$$

Note that φ, a lower case of phi, is also a symbol commonly used for the Golden Ratio.

Since **Φ** is an irrational number, it cannot be realised as a fraction of two integers, therefore it is not pair them as integers. Therefore, approximating those values are the best solutions.

Referring to the segment portion from the beginning of the analysis, when forming the Golden Ratio into a quadratic equation, there is an another interesting characteristic that represents its conjugate.

$$\Phi = \frac{1}{\varphi} = \frac{1}{1.16180339887...} = 0.6180339887$$

This could be alternatively expressed as,

$$\Phi = \varphi - 1 = 1.6180339887\ldots - 1 = 0.6180339887\ldots$$

This property applies among the positive numbers,

$$\frac{1}{\varphi} = \varphi - 1$$

And the inverse is

$$\frac{1}{\varphi} = \varphi + 1$$

Bach creates a textural climax on the measure corresponding to the Golden Ratio. This is achieved through having a harmonic tension with a dominant chord along with a break in the constant sixteenth note examination. Dominant chord is a musical terminology that harmonizes the main note, determining whether if the scale becomes major or minor (sad sounding or happy sounding). It usually creates a satisfying harmony if played, and that is what Bach demonstrated the whole time.

Observing the equation, .61803 is to 1 as 1 to 1.61803. The inverse of .618 helps find the golden mean in a piece of music, either by measure or note. Simplistic mathematics, by multiplying .618 by the number of measures or the number of notes in the piece. This would result to become a golden mean.

Essentially, the formula is:

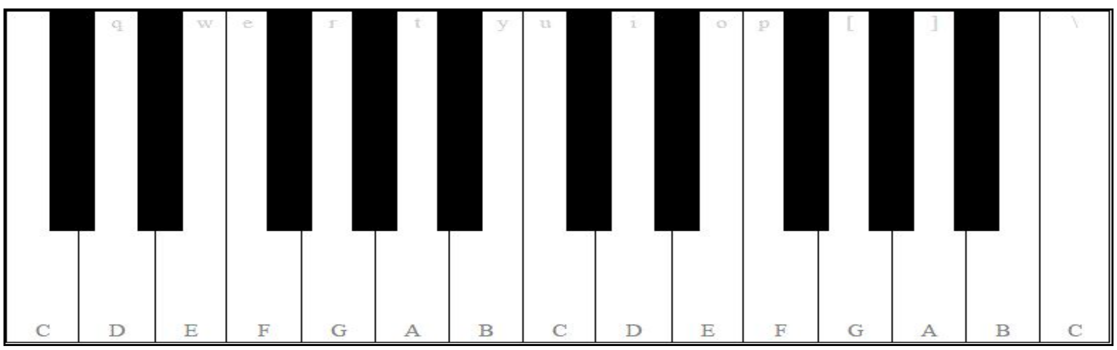
Golden mean = .618 x (The number of measures or notes)

For example, a 56 measure piece would equal to 34. 608, meaning that the golden mean would be located at measure 35 in an approximate way.

With that in mind, Fibonacci sequence is also involved when analyzing the golden ratio of Bach's composition. As the numbers in the series grow larger, the relationship between two numbers next to each other gets closer to the golden ratio. For instance, having series such as the 19th and 20th get 9349 and 15127. Dividing those numbers, it would most likely match the value of the golden ratio, getting 1.618034.

Typing in more information just so I can seem to know what I am doing.

Body:



In music, there are several ways of connecting mathematical concepts as well. For example , there are thirteen notes in one-octave scale. And the most important note is the dominant, which is the fifth note of scale and the third note chromatically. For an octave of C on the piano, there are thirteen notes in total: 8 white keys and 5 black keys respectively. For example, in order to play a musical scale of "C", only the white keys are played. The third note ("E") and the fifth note ("G") creates the basic foundation of chords and determines whether if a chord is a major chord or a minor chord. These numbers are part of the Fibonacci sequence. Because of that, Bach's compositions that have the sequence demonstrates texture changes and harmonic tension.

The eight notes ("C" to "C"), all of the notes and intervals only utilize Fibonacci sequence numbers 1,2,5,8, and 13.

Figure 1. An excerpt from Bach's Prelude No.1

Bach's compositions are known for having a steady pattern for arithmetic and visual purposes, as it is depicted above. It has sixteenth notes, which are the notes that has two black lines crossed on top of the musical notes. This piece for example, has 35 measures total. Doing the math, it equals to 21.63. This indicates that the golden mean is located at measure 22. That part is when a key change occurs. Which indicates as a climax and a note change from measure 22. Recall that earlier, the equation to find the golden ratio, the golden ratio inverse multiplied by the number of measures or notes. Therefore, measure 22 is the measure that also has a rhythmic and a textural climax.

An another example is the *Art of Fugue*. A fugue is a composition in which a short melody is introduced by one part and taken by others which creates an interweaving effect. Bach was in fond of counterpoint while composing fugues, canons and variations. In most Western music, there is a single melodic line, while other parts (notes) add on in order to create harmony. In counterpoint, there are multiple lines, with notes playing a vital role as harmony and melody in a simultaneous manner.

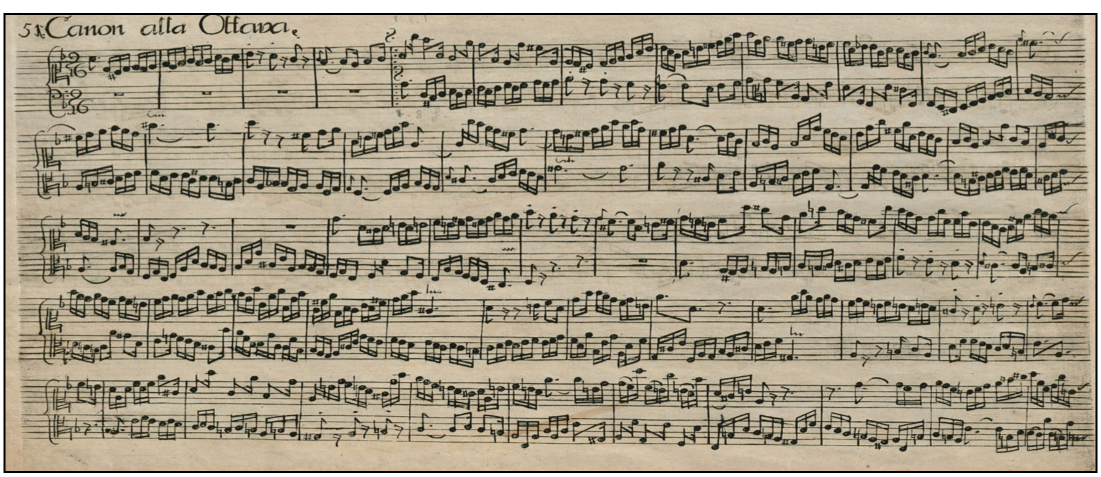


Figure 2. An excerpt from Bach's Contrapunctus.

Notice that there are two lines paired each other. The line on the top is the melody and the one below is the bass part. The melody introduces a phrases, and on the fifth measure, the base part plays the same phrase from the first part that was played. The rhythm and the texture interweaves each other if observed closely. This is essentially an overall view of the structure of Fugue.

Now diving in Bach's *Fugue of Art*, utilizing mathematics, Fibonacci sequence was everywhere. The first fourteen fugues were chosen for proof.

Number	BWV Classification (1080)	Name	Total Bars
1	1	Contrapunctus 1	78
2	2	Contrapunctus 2	84
3	3	Contrapunctus 3	72
4	4	Contrapunctus 4	138
5	5	Contrapunctus 5	90
6	6	Contrapunctus 6	79
7	7	Contrapunctus 7	61
8	8	Contrapunctus 8	188
9	9	Contrapunctus 9	130
	10a		100
10	10	Contrapunctus 10	120
11	11	Contrapunctus 11	184
12	12	Contrapunctus 12	56 (rectus) & 56 (inversus)
13		Contrapunctus 13	56
-		-	71 (rectus) & 71 (inversus)

-		-	109
-		-	103
-	-	-	82
-	-	-	78
14		Contrapunctus 14	239

Figure 3. Chart for contrapunctus and bar numbers to determine Fibonacci Sequence.

Each musical bar has a small amount of time. Most of music - especially Baroque Music - has a beat or a pulse that once could feel. A piece usually consists of having several bars of the same length within the time signature. A contrapunctus is a melodic material that is added above or below the melody. Commonly utilized in fugues, the contrapunctus charts demonstrate the complex mathematics of Fibonacci sequence.

The data was gathered through his compositions, counting his total measures for each fugue. A measure is a one “chunk” that fits notes into a box-like place.
According to the researchers, they have found a numerical symbolism throughout his works. One of them includes numbers derived from the number alphabets, which the alphabets are associated with the number of its position in the alphabet. Known as the “Bach number”, derived from B + A + C + H = 2 + 1 + 3 +8 = 14, it was one of the theories as well.

The *Art of Fugue* reveals a structural sequence. Looking in deeper, the numbers identify the counterpoints from the data provided above. Observing the graph, there are numbers that are the same but has different terminologies: rectus and inversus. Those were not written separately, but rather due to a different structure.
Analyzing fourteen fugues, the first (1-7) and (8-14) were subdivided into two subsets respectively. The first subsets consists of 602 bars and the second subset consisted of 988 bars.

$$\frac{\sum Counterpoints\ 8,9,10,11,12,13,14}{\sum Counterpoints\ 1,2,3,4,5,6,7} = \frac{988}{602} = 1.641..$$

Doing some experimentation, arranging the numbers around through trial and error, there was this:

$$\frac{\sum Counterpoints\ 1,2,3,4,5,6,7,8,9,10,11,12,13,14}{\sum Counterpoints\ 8,9,10,11,12,131,4} = \frac{1590}{988} = 1.609..$$

These values are very close to the number of a golden ratio.

Furthermore, what is even more fascinating is that with canon bars, it creates a perfect golden ratio. A Canon is a piece of voices that plays the same music starting at different times. This happens very often throughout his works. He utilizes complex musical structures to have more than one melody intersecting one and another.

After counting, the total number of bars (Canon) is 372.

$$\frac{\sum Counterpoints\ 1,2,3,4,5,6,7}{\sum Canons} = \frac{602}{372} = 1.618..$$

Notice that there are evidence already figured out by other mathematicians. What makes it special is the correlation of Math and Music merged into as one. The music itself may start as having a defined melody into complex, elaborative notes demonstrating the complexity of mathematics. It is clear that the Fibonacci sequence and the golden ratio are manifested in music, as it is shown above with all the mathematical formulas discovered by mathematicians from the past and myself as well. The numbers are present within the musical notes, which is the foundational unity and harmony. Furthermore, it defines the aesthetic appeal of this mathematical phenomenon throughout the investigation.

Conclusion:

Through these examples, there are creative ways of how mathematics are everywhere. Furthermore, how composers utilized the golden ratio and the Fibonacci sequence as indicates their mathematical skills. Although there is no certainty whether if this was intentional or by chance, however, it still links the minds and compositions of Bach’s music and his touch regarding structure, logic, and proportion. Although it was brief, with some finite examples, I was able to understand the geometrical structure of the Fibonacci sequence and the golden ratio. If I were to have more time to spend more time on this assessment, I would begin to analyze his other works that are fugues and search for Fibonacci sequence. Counting beats while playing a certain piece of music requires simplistic mathematics.Although I was not able to validify the causation of Fibonacci Sequence's formation, I was able to discover the patterns and the numbers within a musical piece. It was an astounding moment for me to realize of how numbers are hidden inside musical notes.

However, knowing that Bach's work reveals not only musical patterns, but also depicting a mathematical pattern. It is fascinating It is true that it was noted for having a diverse usage not only in music, but also for architecture, art, and nature. It was a fascinating experiment for me to not only understand music in a mathematical aspect, but to also have the opportunity to appreciate number theory.

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